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AN IMPROVED PERTURBATION THEORY METHOD FOR DESCRIBING THE EFFECTIVE PERMEABILITY OF A RANDOMLY HETEROGENEOUS MEDIUM†

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Assuming a log-normal statistics of the permeability field of a randomly heterogeneous medium, a equation is obtained for the integral kernel, defining the non-local relation between the values of the gradient of the thermodynamic potential and the flow, averaged over the ensemble of realizations of the permeability. The solution of this equation in the large-scale limit reproduces the well-known Landau–Lifshitz–Matheron formula, and in the general case enables a relation to be obtained between the two-point permeability covariance function of the fluctuations in the permeability and the form of the integral kernel. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

When describing transport phenomena in weakly heterogeneous systems, a linear relation between the gradient of a certain thermodynamic potential, characterizing the degree of heterogeneity of the system and the flow which occurs, is usually used. In an isotropic system this relation has the form

$$J_i(\mathbf{r}) = -\varkappa\nabla_i\varphi(\mathbf{r}) \tag{1.1}$$

where the coefficient of proportionality $\varkappa$ will be called the permeability of the medium. In a randomly heterogeneous medium the permeability is a random function of the coordinates, as a result of that the potentials and the flows also turn out to be random functions of the coordinates.

In practical problems, the relation between the gradients and the flows, averaged over the ensemble of realizations of the permeability, is of direct interest. In the general case this relation is non-local and has the form

$$\langle J_i(\mathbf{r})\rangle = -\int K(\mathbf{r}-\mathbf{r}')\langle\nabla_i\varphi(\mathbf{r}')\rangle d\mathbf{r}' \tag{1.2}$$

where the integral kernel $K(\mathbf{r})$ is non-zero in a certain finite region with dimensions of the order l. If the characteristic dimensions, over which $\langle\nabla_i\varphi(\mathbf{r})\rangle$ varies considerably, are large compared with l (the large-scale limit), we can assume

$$K(\mathbf{r}-\mathbf{r}') = K_{\text{eff}}\delta(\mathbf{r}-\mathbf{r}'), \qquad K_{\text{eff}} = \int K(\mathbf{r})d\mathbf{r} = \tilde{K}(\mathbf{q})|_{q=0}$$
$$K(\mathbf{r}) = \int \tilde{K}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}\frac{d\mathbf{q}}{(2\pi)^d}, \qquad \tilde{K}(\mathbf{q}) = \int K(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r} \tag{1.3}$$

where $\tilde{K}(\mathbf{q})$ is the Fourier transform of the integral kernel $K(\mathbf{r})$ and d is the dimensionality of the space.

The problem consists of calculating the effective permeability K_{eff}, and also the integral kernel $K(\mathbf{r})$ (or its Fourier transform $K(\mathbf{q})$) for a specified statistics of the realizations of the random field $\varkappa(\mathbf{r})$.

We will investigate this problem below using the example of seepage flows in porous media, where the pressure p plays the role the thermodynamic potential. The seepage velocity v_i will correspond to the flow and Darcy's law

$$v_i(\mathbf{r}) = -\varkappa(\mathbf{r})\nabla_i p(\mathbf{r}) \tag{1.4}$$

will correspond to the functional equation of the theory of irreversible processes (1.1).

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We will consider an unbounded medium with a specified regular source of liquid of density $\rho(\mathbf{r})$. It follows from the condition of incompressibility that

$$\nabla_i v_i(\mathbf{r}) = \rho(\mathbf{r}) \tag{1.5}$$

Using Darcy's law, we obtain the equation for the pressure

$$\nabla_i[\varkappa(\mathbf{r})\nabla_i p(\mathbf{r})] = -\rho(\mathbf{r}) \tag{1.6}$$

The main difficulty in solving such problems lies in the fact, to find the averaged characteristics, it is first necessary to construct a solution of the stochastic differential equation in the form of a functional of the random field realization $\varkappa(\mathbf{r})$, and the average this solution over the ensemble of realizations. In general, it is not possible to represent the solution of an equation with variable (random) coefficients in closed form, and an exact stochastic solution can only be obtained in very special cases.

The simplest method of calculating the effective permeability is to use perturbation theory, when the solutions for the seepage velocity and the pressure gradient can be represented in the form of series expansion in powers of the quantity $\delta\varkappa(\mathbf{r})/\langle\varkappa\rangle$, considered as a small parameter. Subsequent term-by-term averaging of the series obtained for a given statistics of the permeability fluctuations (usually normal or log-normal) enables one to calculate $\langle \mathbf{v} \rangle$ and $\langle \nabla_p \rangle$ and thereby obtain the effective permeability in some approximation of perturbation theory in the form of a series in powers of the variance of the permeability fluctuations. Such an approach will be called by convention, "simple perturbation theory" (see [1]). In a similar approach one must confine oneself solely to the lower-order approximations of perturbation theory and the question of how well the lower-order approximations of perturbation theory describe the behaviour of the effective permeability for large variances and the question of the convergence of the series remain open. In connection with estimating the role of higher-order approximations based on an analysis of the exact results for one-dimensional case, the log-normal distribution of fluctuations of the permeability in two-dimensional case [2] and general phenomenological considerations [3], the suggestion was put forward in [4] that the dependence of the effective permeability on the variance of the logarithm of the permeability is exponential, and lower-order approximations of perturbation theory correspond to the Taylor expansion of the exponential function. The corresponding hypothesis has come to be called the Landau–Lifshitz–Matheron formula. Numerous investigations have been carried out over a long period of time to substantiate this formula. In particular, this hypothesis has found confirmation, at least up to second-order terms in the variance [5], and these are indications that it breaks down in higher approximations [6,7].

To improve perturbation theory by summing a certain infinite subsequence of the whole series of perturbation theory, methods borrowed from quantum field theory are used. In particular, instead of collecting terms of the same order of smallness, employed when constructing series within the framework of the simple perturbation theory, a change is made from a differential equation for Green's function to an integral equation, the iterational solution of which reproduces the perturbation theory series [8]. Using this approach, it is easy to investigate the structure of an arbitrary term of the series. the elements of this structure can be uniquely set up using certain graphical symbols – Feynman diagrams, and the series can be analysed later in the language of Feynman diagrams [8–10]. In particular, using this approach it is possible to sum the diagram series and obtain Dyson's equation, which contains a certain new element, called the self-energy operator (as it applies to other problems this is sometimes called the polarization operator). The use of perturbation theory for the self-energy operator, following its substitution into Dyson's equation and the solution of the equation obtained, corresponds to summation of a certain infinite subsequence of the whole series – summation of one-particle irreducible diagrams. This approach is naturally called "partial summation of a perturbation theory series".

A further improvement in perturbation theory can be achieved using the renormalization group (renormgroup) method. This method was apparently used for the first time to investigate processes in random media in [10]. The renormgroup method was used to further improve perturbation theory, when the self-energy operator is determined not in the lowest approximation of perturbation theory a solution of a certain equation corresponding to self-consistent field theory. This approach has been developed further in a number of other publications (see, for example, [11], which contains the most complete review of the renormgroup approach to the problem).

This paper is devoted to investigating the behaviour of the effective permeability using an improved perturbation theory based on partial summation of the complete perturbation-theory series. Unlike the usual method for obtaining the diffusion coefficient approximately using perturbation theory, in the proposed approach an approximate equation is constructed for the integral Kernel $K^{-1}(\mathbf{r})$, defined by the relation

$$\langle \nabla_i p(\mathbf{r})\rangle = -\int K^{-1}(\mathbf{r}-\mathbf{r}')\langle v_i(\mathbf{r}')\rangle d\mathbf{r}' \tag{1.7}$$

Knowing $K^{-1}(\mathbf{r})$ we can calculate the effective permeability from the formula

$$K_{\text{eff}}^{-1} = \int K^{-1}(\mathbf{r})d\mathbf{r} \tag{1.8}$$

2. GOVERNING EQUATIONS

We will represent the velocity field in the form of the superposition of potential and solenoidal parts

$$v_i(\mathbf{r}) = v_i^p(\mathbf{r}) + v_i^s(\mathbf{r}); \qquad \nabla_i v_j^p(\mathbf{r}) = \nabla_j v_i^p 0, \qquad \nabla_i v_i^s(\mathbf{r}) = 0 \tag{2.1}$$

In this case Eq. (1.6) only defines the potential part

$$v_i^p(\mathbf{r}) = (\nabla)_i^{-1}\rho(\mathbf{r}) = \nabla_i \Delta^{-1}\rho(\mathbf{r}) \tag{2.2}$$

where $\Delta^{-1} = G^{(0)}(\mathbf{r})$ is Green's function for the Laplace operator, which is the solution of the equation

$$\Delta G^{(0)}(\mathbf{r}) = \delta(\mathbf{r}) \tag{2.3}$$

In order to find the equation for the solenoidal part of the seepage flow velocity v_i we will calculate a quantity which, in three-dimensional space, corresponds to rot rot $\mathbf{v}(\mathbf{r})$. Taking relations (2.1) and Darcy's law (1.4) into account we obtain

$$\nabla_j[\nabla_i v_j(\mathbf{r}) - \nabla_j v_i(\mathbf{r})] = -\nabla_j \nabla_j v_i^s(\mathbf{r}) = -\nabla_j[\nabla_i \varkappa(\mathbf{r})\nabla_j p(\mathbf{r}) - \nabla_j \varkappa(\mathbf{r})\nabla_i p(\mathbf{r})]$$

Again, using Darcy's law to eliminate the pressure, we obtain

$$\nabla_j \nabla_j v_i^s(\mathbf{r}) = -[\nabla_k u_k(\mathbf{r})\delta_{ij} - \nabla_j u_i(\mathbf{r})]v_j(\mathbf{r}), \qquad u_i(\mathbf{r}) = \nabla_i u(\mathbf{r}) \tag{2.4}$$

where $\varkappa(\mathbf{r}) = \varkappa_0 \exp\{u(\mathbf{r})\}$ and $\varkappa_0$ is an arbitrary constant with the dimension of permeability. If we require that $\langle u(\mathbf{r})\rangle = 0$, the constant $\varkappa_0$ turns out to be equal to the geometric mean

$$K_G = \lim_{N\to\infty}\left[\prod_n^N \varkappa_n\right]^{1/N}$$

As a result, the equation for the components of the seepage velocity flow takes the form

$$v_i(\mathbf{r}) = v_i^p(\mathbf{r}) - \int G^{(0)}(\mathbf{r}-\mathbf{r}')[\nabla_k' u_k(\mathbf{r}')\delta_{ij} - \nabla_j' u_i(\mathbf{r}')v_j(r')d\mathbf{r}' \tag{2.5}$$

It can also be represented in a form corresponding to the Martin–Siggia–Rose formalism [12]

$$v_i(\mathbf{r}) = v_i^p(\mathbf{r}) - \int G^{(0)}(\mathbf{r}-\mathbf{r}_1)\Gamma_{ij}^{(0)}(\mathbf{r}_1 \mid \mathbf{r}_2, \mathbf{r}_3)u(\mathbf{r}_2)v_j(\mathbf{r}_3)d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \tag{2.6}$$

where the function $\Gamma_{ij}^{(0)}$ $(\mathbf{r}_1 \mid \mathbf{r}_2, \mathbf{r}_3)$ (usually called the vertex function in diagram technique) has the form

$$\Gamma_{ij}^{(0)}(\mathbf{r}_1 \mid \mathbf{r}_2, \mathbf{r}_3) = \lambda_0[(\nabla^{(2)} \cdot \nabla^{(1)})\delta_{ij} - \nabla_i^{(2)}\nabla_j^{(1)}]\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{r}_1 - \mathbf{r}_3)$$

where λ_0 is a formal parameter of the expansion in the perturbation theory series, equal to unity, and the superscript parentheses on the differentiation operators indicates on which of the coordinates the operator acts.

The function Γ_{ij} possesses the following properties

$$\nabla_i^{(1)}\Gamma_{ij}(\mathbf{r}_1 \mid \mathbf{r}_2, \mathbf{r}_3) = 0, \qquad \int P_{in}(\mathbf{r}_1 - \mathbf{r}_1')\Gamma_{nj}(\mathbf{r}_1' \mid \mathbf{r}_2, \mathbf{r}_3)d\mathbf{r}_1' = \Gamma_{ij}(\mathbf{r}_1 \mid \mathbf{r}_2, \mathbf{r}_3) \tag{2.7}$$

where $P_{ij}(\mathbf{r}) = \delta_{ij}\delta(\mathbf{r}) - \nabla_i\nabla_j\Delta^{-1}(\mathbf{r})$ is the transverse projection operator.

3. A FORMULA FOR THE EFFECTIVE PERMEABILITY

The use of the method of iterations leads to a representation of the solution of Eq. (2.5) in the form

$$v_i(\mathbf{r}) = \int R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r}))v_j^p(\mathbf{r}')d\mathbf{r}' \tag{3.1}$$

where the resolvent kernel R_{ij} is a functional of the realizations of the random field $u(\mathbf{r})$.

Substituting (3.1) into (2.5), we obtain an equation for R_{ij}

$$\begin{aligned} R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r})) &= \delta_{ij}\delta(\mathbf{r}-\mathbf{r}') - \\ &-\int G^{(0)}(\mathbf{r}-\mathbf{r}_1)u(\mathbf{r}_2)\Gamma_{ik}^{(0)}(\mathbf{r}_1 \mid \mathbf{r}_2,\mathbf{r}_3)R_{kj}(\mathbf{r}_3,\mathbf{r}' \mid u(\mathbf{r}))d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \end{aligned} \tag{3.2}$$

which can be rewritten in the equivalent form

$$\begin{aligned} R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r})) &= \delta_{ij}\delta(\mathbf{r}-\mathbf{r}') - \\ &-\int R_{ik}(\mathbf{r},\mathbf{r}_1 \mid u(\mathbf{r}))G^{(0)}(\mathbf{r}_1-\mathbf{r}_3)u(\mathbf{r}_2)\Gamma_{kj}^{(0)}(\mathbf{r}_3 \mid \mathbf{r}_2,\mathbf{r}')d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \end{aligned} \tag{3.3}$$

Averaging relation (3.1) over the ensemble of realizations, taking into account the statistical homogeneity of the system, we obtain

$$\langle v_i(\mathbf{r})\rangle = \int \overline{R}_{ij}(\mathbf{r}-\mathbf{r}')v_j^p(\mathbf{r}')d\mathbf{r}', \qquad \overline{R}_{ij}(\mathbf{r}-\mathbf{r}') = \langle R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r}))\rangle \tag{3.4}$$

The general structure of the tensor integral kernel $\overline{R}_{ij}(\mathbf{r}-\mathbf{r}')$ has the form

$$\overline{R}_{ij}(\mathbf{r}) = \delta_{ij}\delta(\mathbf{r}) + \int P_{ij}(\mathbf{r}-\mathbf{r}')\overline{R}^{(1)}(\mathbf{r}')d\mathbf{r}'$$

Since in expression (3.4) $\overline{R}_{ij}$ is contracted with the potential vector and the contraction of the tranverse-projection operator with the potential vector is equal to zero, we obtain

$$\langle v_i(\mathbf{r})\rangle = v_i^p(r) = \nabla_i\Delta^{-1}\rho(\mathbf{r}) \tag{3.5}$$

and the problem of calculating the effective permeability thereby reduces to finding $\langle\nabla_i p(\mathbf{r})\rangle$.

To calculate the pressure gradient we will use Darcy's law

$$\nabla_i p(\mathbf{r}) = -\varkappa^{-1}(\mathbf{r})\int R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r}))v_j^p(\mathbf{r}')d\mathbf{r}' \tag{3.6}$$

Averaging relation (3.6), taking (3.5) into account, we obtain a representation for the integral kernel of the inverse permeability

$$K_{ij}^{-1}(\mathbf{r}-\mathbf{r}') = \langle\varkappa^{-1}(\mathbf{r})R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r}))\rangle = K_G^{-1}\langle e^{-u(\mathbf{r})}R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r}))\rangle \tag{3.7}$$

Using (3.2) we obtain the relation

$$\begin{aligned} K_{ij}^{-1}(\mathbf{r}-\mathbf{r}') &= \langle\varkappa^{-1}\rangle\delta_{ij}\delta(\mathbf{r}-\mathbf{r}') - \\ &-K_G^{-1}\int\langle e^{-u(\mathbf{r})}R_{in}(\mathbf{r}-\mathbf{r}_1 \mid u(\mathbf{r}))u(\mathbf{r}_2)\rangle G^{(0)}(\mathbf{r}_1-\mathbf{r}_3)\Gamma_{nj}^{(0)}(\mathbf{r}_3 \mid \mathbf{r}_2,\mathbf{r}')d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \end{aligned} \tag{3.8}$$

In order to obtain an equation for K_{ij}^{-1} we will use a formula, known as the Furutsu–Novikov formula [13,14], that holds for a normal distribution of the field $u(\mathbf{r})$

$$\langle F\{u(\mathbf{r})\}u(\mathbf{r}')\rangle = \left\langle\frac{\delta F\{u(\mathbf{r})\}}{\delta u(\mathbf{r})}\right\rangle B(\mathbf{r}-\mathbf{r}') \tag{3.9}$$

$$B(\mathbf{r}-\mathbf{r}') = \langle u(\mathbf{r})u(\mathbf{r}')\rangle, \qquad \langle u(\mathbf{r})\rangle = 0$$

As it applies to the case (3.8) this gives

$$\begin{aligned} &\langle e^{-u(\mathbf{r})}R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r}))u(\mathbf{r}_2)\rangle = \\ &= -\langle e^{-u(\mathbf{r})}R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r}))\rangle B(\mathbf{r}-\mathbf{r}_2) + \int\left\langle e^{-u(\mathbf{r})}\frac{\delta R_{ij}(\mathbf{r},\mathbf{r}' \mid u(\mathbf{r}))}{\delta u(\mathbf{r}'')}\right\rangle B(\mathbf{r}''-\mathbf{r}_2)d\mathbf{r}'' \end{aligned} \tag{3.10}$$

The integrand here contains a statistical average, to find which we must calculate $\delta R/\delta u$ using the equation for the resolvent (3.3)

$$\frac{\delta R_{ij}(\mathbf{r},\mathbf{r}')}{\delta u(\mathbf{r}'')} = -\int R_{in}(\mathbf{r},\mathbf{r}_1)G^{(0)}(\mathbf{r}_1-\mathbf{r}_1')\Gamma_{nj}(\mathbf{r}_1' \mid \mathbf{r}'',\mathbf{r}')d\mathbf{r}_1 d\mathbf{r}_1' -$$

$$-\int \frac{\delta R_{in}(\mathbf{r},\mathbf{r}_1)}{\delta u(\mathbf{r}'')} G^{(0)}(\mathbf{r}_1-\mathbf{r}_1')\Gamma_{nj}(\mathbf{r}_1' \mid \mathbf{r}_2,\mathbf{r}')u(\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_1' d\mathbf{r}_2 \tag{3.11}$$

By multiplying this expression by $\exp\{-u(\mathbf{r}_1)\}$ and averaging over the ensemble of realizations of the random field $u(\mathbf{r})$ using the Furutsu–Novikov formula, we obtain an equation for $\langle \exp\{-u\}\delta R/\delta u\rangle$ involving an expression of the form $\langle \exp\{-u\}\delta^2 R/(\delta u)^2\rangle$ in addition to K^{-1}, in which the second variational derivative can be obtained by calculating the variational derivative of $\delta R/\delta u$ with subsequent multiplication by $\exp\{-u\}$ and using the Furutsu–Novikov formula. Multiple repetition of this procedure leads to a chain of equations containing all higher-order variational derivatives. The infinite system of equations obtained is analogous to the chain of equations for many-particle distribution functions by Bogolyubov–Born–Green–Ivon–Kirkwood in statistical physics or the chain of Fridman–Keller equations in the theory of turbulence. The chain is usually terminated using some closure scheme.

In the case considered here, a direct analysis of this infinite chain shows that the effect of the term containing $\delta R/\delta u$ in the equation for K^{-1} reduces to replacing Green's function $G^{(0)}$ and the vertex function $\Gamma^{(0)}$ in the second term on the right-hand side of Eq. (3.8) by certain new quantities, which will be called the complete (renormalized) Green's function and the complete (renormalized) vertex function respectively. A consideration of the effects of renormalization of Green's and the vertex function within the framework of the renormalization group method shows that these effects compensate one another. The presence of this compensation was found earlier in [10] when investigating the similar problem of obtaining the effective diffusion coefficient in a random velocity field, and, as it applies to the present problem, this result was obtained within the framework of the renormgroup approach in [15].

When calculating the effective permeability using perturbation theory it has been assumed [16] that the contributions of terms which take into account the renormalization of Green's function and the renormalization of the vertex function are small. However, an examination of these contributions in lower-order approximations of perturbation theory shows that this assumption is not justified, i.e. the individual contributions are not small, but their sum is small. Assuming that the two effects compensate one another we can drop the last term on the right-hand side of Eq. (3.10) and obtain a closed equation for the integral kernel K_{ij}^{-1}

$$K_{ij}^{-1}(\mathbf{r}-\mathbf{r}') = \langle \varkappa^{-1}\rangle\delta_{ij}\delta(\mathbf{r}-\mathbf{r}') +$$

$$+\int K_{in}^{-1}(\mathbf{r}-\mathbf{r}_1)G^{(0)}(\mathbf{r}_1-\mathbf{r}_3)B(\mathbf{r}-\mathbf{r}_2)\Gamma_{nj}^{(0)}(\mathbf{r}_3 \mid \mathbf{r}_2,\mathbf{r}')d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \tag{3.12}$$

Taking into account the relation $K_{ij}^{-1}(\mathbf{r}-\mathbf{r}') = \delta_{ij}K^{-1}(\mathbf{r}-\mathbf{r}')$ and using the explicit form of the operator $\Gamma_{ij}^{(0)}$, we obtain for $\mathbf{r}' = 0$

$$K^{-1}(\mathbf{r}) = \langle \varkappa^{-1}\rangle\delta(\mathbf{r}) + \frac{d-1}{d}\nabla_i B(\mathbf{r})\cdot\nabla_i \int G^{(0)}(\mathbf{r}-\mathbf{r}_1)K^{-1}(\mathbf{r}_1)d\mathbf{r}_1 \tag{3.13}$$

The solution of this equation has the form

$$K^{-1}(\mathbf{r}) = \langle \varkappa^{-1}\rangle\left\{\delta(\mathbf{r}) + \frac{1}{S_d r^{d-1}}\frac{d}{dr}\exp\left[\frac{d-1}{d}[B(r)-B(0)]\right]\right\} \tag{3.14}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the surface of a d-dimensional sphere of unit radius.

However, in order to verify that relation (3.14) is correct we need to take into account that, in view of the singular form of the solutions, all the operators act in the space of generalized functions and must accurately differentiate singular expressions.

We will illustrate this using the example of Green's function of the d-dimensional Laplace equation, which has the form

$$G(\mathbf{r}) = -\frac{1}{(d-2)S_d r^{d-2}}$$

The formal application to this function of the Laplace operator

$$\Delta = \frac{1}{r^{d-1}}\frac{d}{dr}r^{d-1}\frac{d}{dr}$$

gives zero, and not $\delta(\mathbf{r}) = (S_d r^{d-1})^{-1}\delta(r)$, as follows from the equation for Green's function. In order to complete the definition of the action of the Laplace operator on Green's function, we will consider it as the limit of a certain regular function

$$G(\mathbf{r}) = -\lim_{\varepsilon\to+0}\frac{1}{(d-2-\varepsilon)S_d r^{d-2-\varepsilon}}$$

taking, in this case,

$$\delta(\mathbf{r}) = \frac{1}{S_d r^{d-1}}\lim_{\varepsilon\to+0}\frac{\varepsilon}{r^{1-\varepsilon}}$$

The δ-function defined by this equation has the Fourier transform

$$\frac{2\pi^{d/2}\Gamma(\varepsilon/2+1)}{S_d\Gamma(d/2-\varepsilon/2)}\left(\frac{q^2}{4}\right)^{\varepsilon/2}$$

which, in the limit as $\varepsilon \to +0$, is equal to unity, i.e. the Fourier transform of the δ-function.

Using the representation of the solution for $K^{-1}(\mathbf{r})$ in the form

$$K^{-1}(\mathbf{r}) = \frac{\langle\varkappa^{-1}\rangle}{S_d}\lim_{\varepsilon\to\infty}\Delta\int_{r_0}^{r}\exp\left\{\frac{d-1}{d}[B(\eta)-B(0)]\right\}\frac{d\eta}{\eta^{d-1-\varepsilon}} \tag{3.15}$$

it is easy to show that it satisfies Eq. (3.13).

Taking into account the condition $B(\infty) = 0$ and the equality of the quantity $B(0)$ to the log-permeability variance σ^2, we can obtain from expressions (3.14) and (3.15)

$$\frac{1}{K_{\text{eff}}} = \int K^{-1}(\mathbf{r})d\mathbf{r} = \langle\varkappa^{-1}\rangle\exp\left\{-\frac{d-1}{d}\sigma^2\right\} \tag{3.16}$$

In the case of a log-normal distribution of the permeability $\langle\varkappa^{-1}\rangle = \exp\{\sigma^2/2\}/K_G$, that leads to the result

$$K_{\text{eff}} = K_G\exp\left\{\frac{d-2}{2d}\sigma^2\right\} \tag{3.17}$$

which reproduces the Landau–Lifshitz–Matheron formula.

The result (3.14) and (3.15) enables us not only to determine K_{eff}, knowing the log-permeability variance, but also enables us to determine the structure of the integral kernel $K^{-1}(\mathbf{r})$ starting from the form of the correlation function of the permeability fluctuations. To do this we must use the following formula, which holds for a log-normal distribution

$$\exp\{B(|\mathbf{r}-\mathbf{r}'|)-B(0)\} = \frac{\langle\varkappa(\mathbf{r})\varkappa(\mathbf{r}')\rangle}{\langle\varkappa\rangle^3\langle\varkappa^{-1}\rangle} = C(|\mathbf{r}-\mathbf{r}'|) \tag{3.18}$$

As a result, formula (3.15) takes the form

$$K^{-1}(\mathbf{r}) = \frac{\langle\varkappa^{-1}\rangle^{1/d}}{\langle\varkappa\rangle^{1-1/d}}\lim_{\varepsilon\to 0}\Delta\int_{r_0}^{r}[C(\eta)]^{(d-1)/d}\frac{d\eta}{\eta^{d-1-\varepsilon}} \tag{3.19}$$

4. DISCUSSION

We have obtained relations (3.16) and (3.17) above, corresponding to the Laudau–Lifshitz–Matheron formula, in the large-scale limit. However, result (3.15) also enables us to obtain the form of the integral kernel for the inverse effective permeability, knowing the form of the two-point log-permeability covariance. In this connection, the question arises of how correct the derivation of the representation for integral kernel (3.15) is. Note that the change from tensor equation (3.12) to scalar equation (3.13) implies that the tensor $K_{ij}(\mathbf{r})$ is diagonal, i.e. $K_{ij}(\mathbf{r}) = \delta_{ij}C(\mathbf{r})$. In fact, in an isotropic medium the most general form for the second rank tensor has the form

$$K_{ij}^{-1}(\mathbf{r}) = \delta_{ij}C(\mathbf{r}) + \nabla_i\nabla_j\Delta^{-1}C^{(1)}(\mathbf{r}) \tag{4.1}$$

as a result of which Eq. (3.13) can be represented in the form

$$C(\mathbf{r}) + \frac{1}{d}C^{(1)}(r) = \langle\varkappa^{-1}\rangle\delta(\mathbf{r}) + \frac{d-1}{d}\nabla_i B(\mathbf{r})\nabla_i \int G^{(0)}(\mathbf{r}-\mathbf{r}_1)C(\mathbf{r}_1)d\mathbf{r}_1 \tag{4.2}$$

that is identical with Eq. (3.13) only when $C^{(1)}(\mathbf{r}) = 0$.

In order to obtain an equation solely for $C(\mathbf{r})$, we must convolute both sides of Eq. (3.13) with the transverse projection operator, which, after calculating the trace of the tensor expression, gives

$$C(\mathbf{r}) = \langle\varkappa^{-1}\rangle\delta(\mathbf{r}) + \int C(|\mathbf{r}-\mathbf{r}_1|)G^{(0)}(\mathbf{r}_1-\mathbf{r}_3)\bar{\Gamma}(\mathbf{r}_3\,|\,\mathbf{r}_2,\mathbf{r}')d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3$$

$$\bar{\Gamma}(\mathbf{r}_3\,|\,\mathbf{r}_2,\mathbf{r}') = \frac{1}{d-1}\int \Gamma_{in}^{(0)}(\mathbf{r}_3\,|\,\mathbf{r}_2,\mathbf{r})P_{ni}(\mathbf{r}-\mathbf{r}')d\mathbf{r} \neq \frac{1}{d}\Gamma_{ii}^{(0)}(\mathbf{r}_3\,|\,\mathbf{r}_2,\mathbf{r}') \tag{4.3}$$

and only if we assume $C^{(1)}(\mathbf{r}) = 0$ do we obtain Eq. (3.13), the solution of which is given by relations (3.14) and (3.15). Note that when convoluting (4.1) with the potential vector $v_i^p = \nabla_i\varphi$, the contribution of $C^{(1)}$ turns out to be non-zero

$$-\nabla_i p = K_{ij}^{-1}v_j^p = (C + C^{(1)})v_i^p \tag{4.4}$$

$$K_{ij}^{-1} = (C + C^{(1)})\delta_{ij}$$

As a consequence of this, for a more accurate solution of tensor equation (3.12), one should initially determine $C(r)$ using the solution of Eq. (4.3), and then seek $C^{(1)}(\mathbf{r})$ by substituting the expression for $C(\mathbf{r})$ into Eq. (4.2), but in this case it is not possible to obtain the solution in closed form.

Another approximation, used above when closing the chain of equations with functional derivatives for the resolvent kernel, is based on the assumption that it is possible to neglect $\delta R/\delta u$ in Eq. (3.10). The ground for this assumption is calculations carried out using the renormgroup approach, when calculations using perturbation theory are carried out for a certain model containing logarithmic divergences. The results obtained in this way, after renormalization, are summed using the requirement of renormgroup invariance of the complete series of perturbation theory, and these results are continued analytically with respect to a certain parameter to a value corresponding to the actual model. With this approach it in fact turns out that the Fourier transform $C^{(1)}$ does not contain logarithmic divergences, and the contributions of terms containing these divergences, which take into account the renormalization of Green's function and the vertex function, cancel each other out.

To investigate to what extent the results of the renormgroup approach validate the above approximations, we calculated the effective permeability in the second approximation of perturbation theory for the model case $B(r) = B_0 \exp\{-r^2/(4m)\}$. Calculation gives

$$K_{\text{eff}}^{-1} = K_G^{-1}\left[1 + \frac{d-1}{d}\sigma^2 + \frac{d-1}{2d}\sigma^4(a_1 + a_2 + a_3)\right]$$

$$a_1 = 1 - \beta\left(\frac{d}{2}\right),\quad a_2 = -1 + (d-1)\beta\left(\frac{d}{2}\right),\quad a_3 = \frac{d-1}{d} - (d-2)\beta\left(\frac{d}{2}\right),$$

$$\beta\left(\frac{d}{2}\right) = \int_0^1 \frac{t^{d/2-1}}{1+t}dt \tag{4.5}$$

where a_1 is the contribution of the second iteration when solving tensor equation (3.12), and a_2 and a_3 take into account the contributions of the effects of renormalization of Green's function and the vertex function respectively. It follows from expression (4.5) that the assumptions made above turn out to be incorrect at least for $d = 2$, although one also obtains an overall result corresponding to the Landau–Lifshitz–Matheron formula. However, when one takes into account the fact that, as d increases, the function $\beta(d/2) \to 1/d$, we obtain $a_1 \to (d-1)/d$, $a_2 \to -1/d$, $a_3 \to 1/d$, and in the limit of large d the assumptions employed turn out to be correct. Numerical estimates show that when $d = 3$ the error due to the use of these approximations does not exceed 15%, while for the hypothetical case $d = 4$ the error does not exceed 8%. These estimates make us confident that the use of formulae (3.14) and (3.15) to be obtain the form of the integral kernel $K_{ij}^{-1}(\mathbf{r})$ will not lead to considerable errors when constructing the statistical solution of the problem of transport in a randomly inhomogeneous medium.

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